Quantum stochastic differential equations and continuous measurements: unbounded coefficients

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Abstract

A natural formulation of the theory of quantum measurements in continuous time is based on quantum stochastic differential equations (Hudson-Parthasarathy equations). However, such a theory was developed only in the case of Hudson-Parthasarathy equations with bounded coefficients. By using some results on Hudson-Parthasarathy equations with unbounded coefficients, we are able to extend the theory of quantum continuous measurements to cases in which unbounded operators on the system space are involved. A significant example of a quantum optical system (the degenerate parametric oscillator) is shown to fulfill the hypotheses introduced in the general theory.

1 Introduction

A powerful formulation of the quantum theory of measurements in continuous time is based on quantum stochastic calculus [20,21]. In such an approach, the quantum stochastic Schrödinger equation, or Hudson-Parthasarathy equation (HP-equation), is combined with suitable field observables [3,5,7–9]; the resulting formulation is particularly suited for applications in quantum optics

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and for building up a photon detection theory [4,5,11,18,25]. However, the theory is fully developed only for the case in which the HP-equation involves only bounded operators in the initial Hilbert space. Many results are known on the existence and uniqueness of the solution of the HP-equation with unbounded coefficients [13,15,16]; our aim is to combine these results with the equations for the (unbounded) field observables and to show how to arrive to the key evolution equation (40) of the theory of continuous measurements, which concerns the "reduced characteristic operator" (27). Moreover, for applications, it is important to consider the case in which the initial state of the quantum fields is not only the vacuum, but at least a generic coherent vector. This gives that the reduced dynamics is not a quantum dynamical semigroup and we need to handle a master equation with a time-dependent, unbounded Liouville operator. Finally, a relevant physical example is given: the degenerate parametric oscillator [11]. The paper is based on Castro's PhD thesis [12].

For any separable complex Hilbert space $\mathfrak{h}$ let us introduce the following classes of operators on it: $\mathcal{L}(\mathfrak{h})$, the space of bounded linear operators, $\mathcal{U}(\mathfrak{h})$ the class of the unitary operators, $\mathcal{T}(\mathfrak{h})$ the trace-class, $\mathcal{S}(\mathfrak{h}) := \big\{\rho \in \mathcal{T}(\mathfrak{h}) : \rho \geq 0, \operatorname{Tr}\{\rho\} = 1\big\}$ the set of statistical operators.

Then, we introduce the symmetric Fock space $\mathcal{F}$ over $L^2(\mathbb{R}_+;\mathcal{Z})$, where $\mathcal{Z}$ is a d-dimensional complex Hilbert space (the *multiplicity space*) in which we fix a complete orthonormal system $\{z_i,\ i = 1,\ldots,d\}$. We denote by $e(f)$ the *exponential vector* in the Fock space $\mathcal{F}$ associated with the test function $f \in L^2(\mathbb{R}_+;\mathcal{Z})$ and we call *coherent vector* $\psi(f) := \|e(f)\|^{-1} e(f)$. Recall that $\langle e(g)|e(f)\rangle = \exp\langle g|f\rangle$. We assume familiarity with such notions and with quantum stochastic calculus [21]. We shall use the notation $f_i(t) := \langle z_i|f(t)\rangle$ for all $i \geq 1$ and we set $f_0(t) = 1$. We fix the sets

$$\mathcal{M} = L^2(\mathbb{R}_+;\mathcal{Z}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{Z}) \qquad \text{and} \qquad \mathcal{E} = \text{linear span of } \{e(f) : f \in \mathcal{M}\}.$$

The set $\mathcal{E}$ is dense in $\mathcal{F}$ [21, Corollary 19.5 p. 127].

An important feature of the Fock space $\mathcal{F}$ is its structure of continuous tensor product. For any choice of the times $0 \leq s \leq t$ let us introduce the symmetric Fock space $\mathcal{F}_{(s,t)}$ over $L^2((s,t);\mathcal{Z})$ and the symmetric Fock space $\mathcal{F}_{(t}$ over $L^2((t,\infty);\mathcal{Z})$. Then, we have the natural identifications

$$\mathcal{F} \simeq \mathcal{F}_{(0,s)} \otimes \mathcal{F}_{(s,t)} \otimes \mathcal{F}_{(t} \qquad \text{and} \qquad e(f) \simeq e(f_{(0,s)}) \otimes e(f_{(s,t)}) \otimes e(f_{(t})\,, \tag{1}$$

where $f_{(s,t)}(x) := 1_{(s,t)}(x)f(x)$ and $f_{(t}(x) := 1_{(t,\infty)}(x)f(x)$. The symbol $\otimes$ denotes the tensor product of Hilbert spaces, vectors and operators; the algebraic tensor product of dense spaces is denoted by $\odot$.

The *Weyl operator* [21] $W(g;U)$, with $g \in L^2(\mathbb{R}_+;\mathcal{Z})$ and $U \in \mathcal{U}\big(L^2(\mathbb{R}_+;\mathcal{Z})\big)$, is the unique unitary operator on $\mathcal{F}$ defined by

$$W(g;U)\, e(f) = \exp\Big\{-\frac{1}{2}\|g\|^2 - \langle g|\ Uf\rangle\Big\}\, e(Uf+g), \qquad \forall f \in L^2(\mathbb{R}_+;\mathcal{Z}). \tag{2}$$

From the definition one obtains the relations

$$W(g;U)^{-1} = W(g;U)^* = W(-U^*g;U^*)$$

and the composition law

$$W(h;V)W(g;U) = \exp\big\{-\mathrm{i}\,\mathrm{Im}\langle h|Vg\rangle\big\}W(h+Vg;VU). \tag{3}$$

Finally, we denote by $A_i^\dagger(t)$, $\Lambda_{ij}(t)$, $A_j(t)$ the creation, gauge and annihilation processes associated with the basis $\{z_i,\ i=1,\ldots,d\}$; we shall use also the notation

$$\Lambda_{i0}(t) = A_i^\dagger(t), \quad \Lambda_{0j}(t) = A_j(t), \quad \Lambda_{00}(t) = t, \qquad i,j = 1,\ldots,d. \tag{4}$$

In particular we have $\langle e(g)|\Lambda_{ij}(t)e(f)\rangle = \int_0^t \overline{g_i(s)}\, f_j(s)\mathrm{d}s\,\langle e(g)|e(f)\rangle$, $i,j = 0,\ldots,d$.

Let $\mathcal{H}$ be a separable complex Hilbert space, the *initial space*, and let us call $S_\mathcal{H}$ the quantum system with Hilbert space $\mathcal{H}$.

We refer to [15,21] for the definition of quantum stochastic integrals with respect to the operator noises Λ_{ij}, but we need to report at least the notions of adapted process and stochastic integrability.

Definition 1 ([21, p. 180], [15, Definition 2.1])**.** Let D be a dense manifold in $\mathcal{H}$. A family $\{L(t), t\geq 0\}$ of operators in $\mathcal{H}\otimes\mathcal{F}$ is an *adapted process* with respect to $(D,\mathcal{M})$ if (i) $D\odot\mathcal{E}\subset\bigcap_{t\geq 0}\mathrm{Dom}(L(t))$, (ii) the map $t\mapsto L(t)u\otimes e(f)$ is strongly measurable, $\forall u\in D$, $f\in\mathcal{M}$, (iii) $L(t)u\otimes e(f_{(0,t)})\in\mathcal{H}\otimes\mathcal{F}_{(0,t)}$ and $L(t)u\otimes e(f) = \big(L(t)u\otimes e(f_{(0,t)})\big)\otimes e(f_{(t}), \forall t\geq 0, u\in D, f\in\mathcal{M}$.

If additionally the map $t\mapsto L(t)u\otimes e(f)$ is continuous for every $u\in D$ and $f\in\mathcal{M}$ the process is said to be *regular adapted*. Moreover, the adapted process L is said to be *stochastically integrable* if, for all $t\geq 0$, $u\in D$ and $f\in\mathcal{M}$, one has $\int_0^t \|L(s)u\otimes e(f)\|^2\mathrm{d}s < \infty$.

A key notion in the construction of dilations of quantum dynamical semigroups is the one of cocycle [1]. We introduce the strongly continuous one-parameter semigroup $\{\theta(t), t\geq 0\}$ of the shift operators on $\mathrm{L}^2(\mathbb{R}_+;\mathcal{Z})$ and its second quantisation Θ on $\mathcal{F}$: for every $t\geq 0$

$$(\theta_t f)(x) = f(x+t) \qquad \text{and} \qquad \Theta_t e(f) = e(\theta_t f), \qquad \forall f\in\mathrm{L}^2(\mathbb{R}_+;\mathcal{Z}). \tag{5}$$

Let us note that, for $r<s$, $\left(\theta_t 1_{(r,s)}\right)(x) = 1_{(r,s)}(x+t) = 1_{(r-t,s-t)}(x)$; this implies

$$\Theta_t e\big(f_{(0,s)}\big) = e(0), \quad \text{for } 0<s\leq t, \qquad \Theta_t\mathcal{F}_{(r,s)}\subset\mathcal{F}_{(r-t,s-t)} \quad \text{for } 0\leq t\leq r<s.$$

Moreover, it turns out that Θ_t^* is an isometry. We extend Θ_t to the space $\mathcal{H}\otimes\mathcal{F}$ by stipulating that it acts as the identity on $\mathcal{H}$.

Definition 2 (Right and left cocycles)**.** A bounded, adapted operator process $X(t)$ in $\mathcal{H}\otimes\mathcal{F}$ is called *right cocycle* (respectively, *left cocycle*) if for every $s,t\geq 0$ we have $X(t+s) = \Theta_s^* X(t)\Theta_s X(s)$ $\big(X(t+s) = X(s)\Theta_s^* X(t)\Theta_s\big)$.

2 The Hudson-Parthasarathy equations

Let us consider the quantum stochastic differential equation (QSDE) for operators on $\mathcal{H} \otimes \mathcal{F}$, known as *right* HP-equation: $U(0) = \mathbb{1}$,

$$\mathrm{d}U(t) = \Big(\sum_{i \geq 1} R_i \mathrm{d}A_i^\dagger(t) + \sum_{i,j \geq 1} F_{ij} \mathrm{d}\Lambda_{ij}(t) + \sum_{j \geq 1} N_j \mathrm{d}A_j(t) + K \mathrm{d}t \Big) U(t), \qquad (6)$$

where the coefficients $K,\ R_i,\ N_i,\ F_{ij}$, with $i, j = 1, \ldots, d$, are (possibly unbounded) operators in the initial space $\mathcal{H}$. Very general sufficient conditions, which guarantee the existence of a unique solution of (6) and the fact that such a solution is a *unitary cocycle*, are given by Fagnola and Wills [16].

By using the notation (4) and by setting

$$F_{00} = K\,, \quad F_{i0} = R_i\,, \quad F_{0j} = N_j\,, \qquad (7)$$

we can write the right HP-equation in the shortened form

$$\mathrm{d}U(t) = \sum_{i,j \geq 0} F_{ij}\, \mathrm{d}\Lambda_{ij}(t)\, U(t), \qquad U(0) = \mathbb{1}. \qquad (8)$$

We shall need also the adjoint equation, the *left HP-equation*:

$$\mathrm{d}V(t) = V(t) \sum_{i,j \geq 0} F_{ji}^*\, \mathrm{d}\Lambda_{ij}(t), \qquad V(0) = \mathbb{1}. \qquad (9)$$

Definition 3 (Right Solution – [15, Definition 3.2])**.** Let D be a dense subspace in $\mathcal{H}$. An operator process U is a *solution of the right HP-equation* in $D \odot \mathcal{E}$ for the matrix F if:

(i) each operator $F_{ij} \otimes \mathbb{1}$ is closable and $\bigcup_{t \geq 0} U(t)(D \odot \mathcal{E}) \subset \bigcap_{i,j \geq 0} \mathrm{Dom}(\overline{F_{ij} \otimes \mathbb{1}})$;

(ii) each process $\overline{F_{ij} \otimes \mathbb{1}}\ U$ is stochastically integrable and

$$U(t) = \mathbb{1} + \sum_{i,j \geq 0} \int_0^t \overline{F_{ij} \otimes \mathbb{1}}\ U(s)\, \mathrm{d}\Lambda_{ij}(s) \qquad \text{on } D \odot \mathcal{E}, \quad \forall t \geq 0.$$

Definition 4 (Left Solution – [15, Definition 3.1])**.** Let $\widetilde{D}$ be a dense subspace in $\mathcal{H}$. An operator process V is a *solution of the left HP-equation* in $\widetilde{D} \odot \mathcal{E}$ for the matrix F^* if:

(i) $\widetilde{D} \subset \bigcap_{i,j \geq 0} \mathrm{Dom}(F_{ij}^*)$ and the linear manifold $\left(\bigcup_{i,j \geq 0} F_{ij}^* \left(\widetilde{D} \right) \right) \odot \mathcal{E}$ is contained in the domain of $V(t)$, $\forall t \geq 0$;

(ii) the processes $\big(V(t) F_{ij}^*; t \geq 0\big)$ are stochastically integrable and

$$V(t) = \mathbb{1} + \sum_{i,j \geq 0} \int_0^t V(s) F_{ji}^*\, \mathrm{d}\Lambda_{ij}(s) \qquad \text{on } \widetilde{D} \odot \mathcal{E}, \quad \forall t \geq 0.$$

Hypothesis 1. (The matrix F)

(i) $F = (F_{ij}; 0 \le i,j \le d)$ is a matrix of closed operators in the initial space $\mathcal{H}$. By F^* we denote the adjoint matrix, defined by $(F^*)_{ij} = F^*_{ji}$. We also define $\mathrm{Dom}(F) := \bigcap_{i,j\ge 0} \mathrm{Dom}(F_{ij})$, $\quad \mathrm{Dom}(F^*) := \bigcap_{i,j\ge 0} \mathrm{Dom}(F^*_{ji})$

(ii) For $1 \le i,j \le d$, we have $F_{ij} = S_{ij} - \delta_{ij}\mathbb{1}$, where the S_{ij} are bounded operators on $\mathcal{H}$ satisfying the unitarity conditions $\sum_{k=1}^d S^*_{ki}S_{kj} = \sum_{k=1}^d S_{ik}S^*_{jk} = \delta_{ij}$.

(iii) There exist a dense subspace D which is a core for K, R_i, N_i, $i = 1,\dots,d$, and a dense subspace $\widetilde{D}$ which is a core for K^*, R^*_i, N^*_i, $i = 1,\dots,d$.

(iv) $\mathrm{Dom}(N^*_i) \supset D \cup \widetilde{D}$, $\mathrm{Dom}(R_i) \supset D \cup \widetilde{D}$, $\mathrm{Dom}(N_i) \supset \mathrm{Dom}(K)$, $\forall i \ge 1$.

(v) $\forall k \ge 1$, $\forall u \in \mathrm{Dom}(K)$: $S_{ki}u \in \mathrm{Dom}(R^*_k)$, $\forall i \ge 1$.

(vi) The operators K and K^* are the infinitesimal generators of two strongly continuous contraction semigroups on $\mathcal{H}$. Moreover, we have $\forall u \in D$, $\forall v \in \widetilde{D}$

$$2\mathrm{Re}\langle Ku|u\rangle = -\sum_{k\ge 1} \|R_k u\|^2, \qquad 2\mathrm{Re}\langle K^*v|v\rangle = -\sum_{k\ge 1} \|N^*_k v\|^2. \tag{10}$$

(vii) $N^*_i u = -\sum_{k\ge 1} S^*_{ki} R_k u\,, \qquad \forall u \in D \cup \widetilde{D}\,, \quad \forall i \ge 1.$

(viii) There exist a positive self-adjoint operator C on $\mathcal{H}$ and the constants $\delta > 0$ and b_1, $b_2 \ge 0$ such that [16, pp. 281–291] $\mathrm{Dom}(C^{1/2}) \subset \mathrm{Dom}(F)$ and

- **(a)** for each $\epsilon \in (0,\delta)$, there exists a dense subspace $D_\epsilon \subset \widetilde{D}$ such that $C_\epsilon^{1/2} D_\epsilon \subset \widetilde{D}$ and each operator $F^*_{ij} C_\epsilon^{1/2}|_{D_\epsilon}$ is bounded, where $C_\epsilon = \frac{C}{(1+\epsilon C)^2}$;
- **(b)** for all $0 < \epsilon < \delta$ and $u_0, \dots, u_d \in \mathrm{Dom}(F)$, the following inequality holds:

$$\sum_{i,j\ge 0} \left(\langle u_i | C_\epsilon F_{ij} u_j\rangle + \langle F_{ji} u_i | C_\epsilon u_j\rangle + \sum_{k\ge 1} \langle F_{ki} u_i | C_\epsilon F_{kj} u_j\rangle \right) \le \sum_{i\ge 0} \left(b_1 \langle u_i | C_\epsilon u_i\rangle + b_2 \|u_i\|^2 \right).$$

Proposition 1. *Under Hypothesis 1 also the following properties hold:*

1. $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $\mathrm{Dom}(N^*_k) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $k \ge 1$.
2. $\mathrm{Dom}(F) = \mathrm{Dom}(K)$; *Eqs.* (10) *hold* $\forall u \in \mathrm{Dom}(K)$, $\forall v \in \mathrm{Dom}(K^*)$.

3. *$N_i^* u = -\sum_{k\geq 1} S_{ki}^* R_k u$, $R_i u = -\sum_{k\geq 1} S_{ik} N_k^* u$, $\forall u \in \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $\forall i \geq 1$.*

4. *$N_i u = -\sum_{k\geq 1} R_k^* S_{ki} u\,, \qquad \forall u \in \mathrm{Dom}(K), \quad \forall i = 1, \ldots, d.$*

5. *for every choice of $u_0, u_1, \ldots, u_d$ in $\mathrm{Dom}(F)$ and of $v_0, v_1, \ldots, v_d$ in $\widetilde{D}$, we have*

$$\sum_{i,j\geq 0} \left(\langle u_i | F_{ij} u_j \rangle + \langle F_{ji} u_i | u_j \rangle + \sum_{k\geq 1} \langle F_{ki} u_i | F_{kj} u_j \rangle \right) = 0. \tag{11a}$$

$$\sum_{i,j\geq 0} \left(\langle v_i | F_{ji}^{\,*} v_j \rangle + \langle F_{ij}^{\,*} v_i | v_j \rangle + \sum_{k\geq 1} \langle F_{ik}^{\,*} v_i | F_{jk}^{\,*} v_j \rangle \right) = 0. \tag{11b}$$

Proof. By (10) we get, $\forall \phi \in D$, $\|R_k \phi\|^2 \leq 2 |\langle K\phi | \phi \rangle|$. For any $u \in \mathrm{Dom}(K)$ we can find a sequence $u_n \in D$ converging to u. Then, $K u_n \to K u$ weakly and by the proposition at p. 112 of [23] the sequence $K u_n$ is norm bonded: $\|K u_n\| \leq c$. Then, we have $\|R_k(u_n - u_m)\|^2 \leq 2 |\langle K(u_n - u_m) | u_n - u_m \rangle| \leq 4c \|u_n - u_m\|$. Therefore, $R_k u_n$ is a Cauchy sequence. Being R_k closed, $(u, \lim_n R_k u_n)$ belongs to the graph of R_k and, so, $u \in \mathrm{Dom}(R_k)$; this gives $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K)$. By the same property of closed operators, we get that the first equation in (10) can be extended to the whole $\mathrm{Dom}(K)$. Similarly, from the second equation in (10) we get that it can be extended to the whole $\mathrm{Dom}(K^*)$ and that $\mathrm{Dom}(N_k^*) \supset \mathrm{Dom}(K^*)$. Once again by the property above of closed operators and by the unitarity of the operator matrix S, we get that point (vii) can be extended to $\mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$. Therefore, on the same domain, $\sum_{k\geq 1} \|R_k u\|^2 = \sum_{k\geq 1} \|N_k^* u\|^2$. By exchanging R_k and N_k^* in the two equations in (10) we get also $\mathrm{Dom}(R_k) \supset \mathrm{Dom}(K^*)$ and $\mathrm{Dom}(N_k^*) \supset \mathrm{Dom}(K)$. By these results and the definition of $\mathrm{Dom}(F)$ we get $\mathrm{Dom}(F) = \mathrm{Dom}(K)$. This ends the proof of points (1)–(3).

By using the first equation in point (3) of this proposition and point (v) in Hypothesis 1 we have, $\forall u \in \mathrm{Dom}(K)$, $\forall v \in \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$,

$$\langle v | N_i u \rangle = \langle N_i^* v | u \rangle = -\sum_{k\geq 1} \big\langle S_{ki}^* R_k v \big| u \big\rangle = -\sum_{k\geq 1} \big\langle R_k v \big| S_{ki} u \big\rangle = -\sum_{k\geq 1} \big\langle v \big| R_k^* S_{ki} u \big\rangle.$$

By the density of $\mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$ in $\mathcal{H}$ we have the statement in point (4). Equations in point (6) are by direct verification. □

Note the difference in the domains of Eqs. (11). The last requirement in point (iv) has been added just to have Eq. (11a) on the whole $\mathrm{Dom}(F)$ and not only on D. As one can check, also Hypothesis **HGC** at p. 205 of [15] holds for the coefficients of the left HP-equation (9). We collect in the following theorem many results.

Theorem 2 ([16, Proposition 2.2, Theorem 2.3]; [15, Proposition 6.3, Theorems 8.4, 8.5])**.** *Under Hypothesis 1 the left HP-equation* (9) *has a unique solution* $\{V(t);\, t \geq 0\}$ *on* $\widetilde{D} \odot \mathcal{E}$*, which is a strongly continuous left cocycle of contractions. Moreover,* $U(t) = V(t)^*$ *is a right cocycle and solves the right HP-equation on* $\mathrm{Dom}(C^{1/2}) \odot \mathcal{E}$*.*

As we take $U = V^*$, if V is an isometry process, U is a coisometry process and vice versa. The following Proposition is a small variation of Corollary 2.4 of [16] or of Corollary 11.2 of [15].

Proposition 3. *Under Hypothesis 1 the contractive solution* U *of* (8) *introduced in Theorem 2 is a strongly continuous isometric process. Moreover, if* U *is unitary, it is the unique bounded solution on* $\mathrm{Dom}(C^{1/2}) \odot \mathcal{E}$ *of* (8)*.*

Proof. Let $\widetilde{U}$ be another bounded solution and apply the second fundamental formula of QSC ([21, Proposition 25.2], [15, Eq. (2)]) to U, $\widetilde{U}$. We get, $\forall f, g \in \mathcal{M}$, $\forall u, v \in \mathrm{Dom}(C^{1/2})$,

$$\begin{aligned}
&\big\langle \widetilde{U}(t)v \otimes e(g) \big| U(t)u \otimes e(f) \big\rangle - \big\langle v \otimes e(g) \big| u \otimes e(f) \big\rangle \\
&\qquad = \sum_{i,j\geq 0} \int_0^t \mathrm{d}s \; \overline{g_i(s)} \Big\{ \big\langle \widetilde{U}(s)v \otimes e(g) \big| F_{ij} \otimes \mathbb{1}\, U(s)u \otimes e(f) \big\rangle \\
&\qquad\qquad + \big\langle F_{ji} \otimes \mathbb{1}\, \widetilde{U}(s)v \otimes e(g) \big| U(s)u \otimes e(f) \big\rangle \\
&\qquad\qquad + \sum_{k\geq 1} \big\langle F_{ki} \otimes \mathbb{1}\, \widetilde{U}(s)v \otimes e(g) \big| F_{kj} \otimes \mathbb{1}\, U(s)u \otimes e(f) \big\rangle \Big\} f_j(s)\,.
\end{aligned}$$

But U and $\widetilde{U}$ are solutions on $\mathrm{Dom}(C^{1/2}) \odot \mathcal{E}$; by point (i) of Definition 3 and Eq. (11a), we have that the integrand vanishes. Therefore, $\forall f, g \in \mathcal{M}$, $\forall u, v \in \mathrm{Dom}(C^{1/2})$, $\big\langle \widetilde{U}(t)v \otimes e(g) \big| U(t)u \otimes e(f) \big\rangle = \big\langle v \otimes e(g) \big| u \otimes e(f) \big\rangle$ and this is equivalent to $\widetilde{U}(t)^* U(t) = \mathbb{1}$. This equation for $\widetilde{U} = U$ gives that $U(t)$ is isometric, while for U unitary gives $\widetilde{U}(t)^* = U(t)^*$. Being the adjoint of a strongly continuous process, U is weakly continuous and, being an isometry, it is strongly continuous. □

3 Quantum dynamical semigroups and unitary cocycles

For the physical interpretation of the evolution operator $U(t)$ we need it to be a strongly continuous cocycle of unitary operators (see [5] Section 2.2 and references there in). The unitarity is associated to some property of a related quantum dynamical semigroup (QDS); so, we start with some notions on QDSs.

Definition 5 (QDS)**.** Let us consider a family $\{\mathcal{T}(t),\, t \geq 0\}$ of bounded operators on $\mathcal{L}(\mathcal{H})$ with the following properties:

(i) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$, $\forall s, t \geq 0$, and $\mathcal{T}(0)$ is the identity map;

(ii) $\mathcal{T}(t)$ is completely positive, $\forall t \geq 0$;

(iii) $\mathcal{T}(t)$ is a σ-weakly continuous operator on $\mathcal{L}(\mathcal{H})$, $\forall t \geq 0$;

(iv) for each $X \in \mathcal{L}(\mathcal{H})$ the map $t \mapsto \mathcal{T}(t)[X]$ is continuous with respect to the σ-weak topology of $\mathcal{L}(\mathcal{H})$.

Then, the family of operators $\mathcal{T}(t)$ is called a *quantum dynamical semigroup*. If also $\mathcal{T}(t)[\mathbb{1}] = \mathbb{1}$ holds $\forall t \geq 0$, the QDS $\mathcal{T}(t)$ is said to be *Markov* or *conservative*.

Theorem 4 ([13, Theorem 3.22, Corollary 3.23])**.** *Let A be the infinitesimal generator of a strongly continuous contraction semigroup in $\mathcal{H}$ and let L_k, $k = 1, \ldots$, be operators in $\mathcal{H}$ such that the domain of each operator L_k contains the domain of A and for every $u \in \mathrm{Dom}(A)$ we have $2\,\mathrm{Re}\langle u|Au\rangle + \sum_{k\geq 1} \|L_k u\|^2 = 0$.*

For all $X \in \mathcal{L}(\mathcal{H})$, let us consider the quadratic form $\mathcal{L}[X]$ in $\mathcal{H}$ with domain $\mathrm{Dom}(A) \times \mathrm{Dom}(A)$ given by

$$\langle v|\mathcal{L}[X]u\rangle = \langle v|XAu\rangle + \langle Av|Xu\rangle + \sum_{k\geq 1} \langle L_k v|XL_k u\rangle. \tag{12}$$

Then, there exists a QDS $\mathcal{T}(t)$ solving the equation

$$\langle v|\mathcal{T}(t)[X]u\rangle = \langle v|Xu\rangle + \int_0^t \big\langle v\big|\mathcal{L}\big[\mathcal{T}(s)[X]\big]u\big\rangle \mathrm{d}s \tag{13}$$

with the property that $\mathcal{T}(t)[\mathbb{1}] \leq \mathbb{1}$, $\forall t \geq 0$, and such that for every σ-weakly continuous family $\mathcal{T}'(t)$ of positive maps on $\mathcal{L}(\mathcal{H})$ satisfying Eqs. (12) *and* (13) *we have $\mathcal{T}(t)[X] \leq \mathcal{T}'(t)[X]$, $\forall t \geq 0$, for all positive $X \in \mathcal{L}(\mathcal{H})$.*

If moreover the QDS $\mathcal{T}(t)$ is conservative, then it is the unique σ-weakly continuous family of positive maps on $\mathcal{L}(\mathcal{H})$ satisfying Eq. (13).

The QDS $\mathcal{T}(t)$ defined in Theorem 4 is called the *minimal quantum dynamical semigroup* generated by A and L_k, $k = 1, \ldots$. Sufficient conditions to assure Markovianity of a QDS are known [13,15]. In the application we shall use the following result.

Theorem 5 ([15, Theorem 9.6])**.** *Let A, L_k be as in Theorem 4 and suppose that there exist two positive self-adjoint operators Q and Z in $\mathcal{H}$ with the following properties:*

- $\mathrm{Dom}(A)$ *is contained in* $\mathrm{Dom}(Q^{\frac{1}{2}})$ *and is a core for* $Q^{\frac{1}{2}}$*;*
- *the linear manifold* $\bigcap_{k\geq 1} L_k\big(\mathrm{Dom}(A^2)\big)$ *is contained in* $\mathrm{Dom}(Q^{\frac{1}{2}})$*;*
- $\mathrm{Dom}(A) \subset \mathrm{Dom}(Z^{\frac{1}{2}})$ *and*

$$-2\mathrm{Re}\langle u|Au\rangle = \sum_{k\geq 1} \|L_k u\|^2 = \|Z^{\frac{1}{2}} u\|^2, \qquad \forall u \in \mathrm{Dom}(A);$$

- $\mathrm{Dom}(Q) \subset \mathrm{Dom}(Z)$ *and for all* $u \in \mathrm{Dom}(Q^{\frac{1}{2}})$ *we have* $\|Z^{\frac{1}{2}}u\| \leq \|Q^{\frac{1}{2}}u\|$*;*
- *there is a positive constant* b *depending only on* A*,* L_k*,* Q *such that, for all* $u \in \mathrm{Dom}(A^2)$*, the following inequality holds*

$$2\,\mathrm{Re}\langle Q^{\frac{1}{2}}u \big| Q^{\frac{1}{2}}Au\rangle + \sum_{k\geq 1} \|Q^{\frac{1}{2}}L_k u\|^2 \leq b\|Q^{\frac{1}{2}}u\|^2.$$

Then, the minimal quantum dynamical semigroup associated to A *and* L_k *is Markov.*

Now, we can go back to the problem of the unitarity of $U(t)$.

Theorem 6 ([15, Theorems 10.2, 10.3])**.** *Under Hypothesis 1 the contractive left cocycle* V *solving* (9) *is such that the family of operators* $\widetilde{\mathcal{T}}(t)$ *defined by*

$$\langle v|\widetilde{\mathcal{T}}(t)[X]u\rangle = \langle V(t)v \otimes e(0)|(X \otimes \mathbb{1})V(t)u \otimes e(0)\rangle, \quad \forall u, v \in \mathcal{H}, \quad \forall X \in \mathcal{L}(\mathcal{H}).$$

is the minimal QDS generated by K^* *and* N_k^**,* $k = 1, \ldots, d$*.*

Moreover, the following conditions are equivalent:

(ii) *the process* V *is an isometry;*

(ii) *the minimal QDS associated with* K^* *and* N_k^* *is conservative;*

Hypothesis 2 (Markov condition)**.** The minimal QDS generated by K^* and N_k^*, $k = 1, \ldots, d$, is conservative.

Now, both $U(t)$ and $V(t) = U(t)^*$ are isometries and, so, they are unitary operators.

Corollary 7. *Under Hypotheses 1 and 2 the process* U *introduced in Theorem 2 is unitary and it is the unique bounded solution on* $\mathrm{Dom}(C^{1/2}) \odot \mathcal{E}$ *of* (8)*.*

Remark 1. When $U(t)$ is a strongly continuous unitary right cocycle, one can define the unitary evolution

$$U(t,s) := U(t)U(s)^*, \qquad t \geq s \geq 0. \tag{14}$$

It is easy to check that it is strongly continuous in t and s and such that

$$U(t,s) = \Theta_s^* U(t-s)\Theta_s\,, \qquad U(t,r) = U(t,s)U(s,r), \qquad 0 \leq r \leq s \leq t\,. \tag{15}$$

Moreover, the operator $U(t,s)$ is adapted to $\mathcal{H} \otimes \mathcal{F}_{(s,t)}$ in the sense that it acts as the identity on $\mathcal{F}_{(0,s)} \otimes \mathcal{F}_{(t}$ and leaves $\mathcal{H} \otimes \mathcal{F}_{(s,t)}$ invariant. The unitary operator $U(t,s)$ is interpreted as the evolution operator of system $S_{\mathcal{H}}$ and fields (described in the Fock space $\mathcal{F}$) in the interaction picture with respect to the free dynamics of the fields [5,17].

4 Observables and instruments

Let us consider now the case in which we are interested in the behaviour of the system $S_{\mathcal{H}}$, but any action on it is mediated by some input/output fields, for instance the electromagnetic field, represented in the Fock space $\mathcal{F}$. In this situation we can measure only field observables, from which we make inferences on $S_{\mathcal{H}}$; we can speak of an indirect measurement. By choosing field observables which commute also at different times in the Heisenberg picture, we can represent also measurements in continuous time. To this end we show how to construct such observables and how to eliminate the fields by a partial trace; in this way we obtain a description of the continuous measurement in terms of quantities (instruments) related to the system $S_{\mathcal{H}}$ alone [5,7].

To give the observables at all times, we have to give infinitely many commuting selfadjoint operators or their joint spectral measure; the easiest way to do this is to work with the "Fourier transform" of such a spectral measure, the *characteristic operator*. The construction below involves the Weyl operators (2).

Hypothesis 3 (Elements of the characteristic operator)**.** Let $B^1, B^2, \ldots, B^m$ be commuting selfadjoint operators on $\mathcal{Z}$ and let us take $c \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$, $b \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{Z})$ and $h^\alpha \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{Z})$, $\alpha = 1,\ldots,m$, such that

$$\operatorname{Im}\langle h^\alpha(t)|h^\beta(t)\rangle = 0\,, \qquad B^\alpha\, h^\beta(t) = 0\,, \qquad \forall t \geq 0, \quad \forall \alpha,\beta = 1,\ldots,m. \tag{16}$$

Definition 6 (The characteristic operator)**.** For any *test function* $k \equiv (k_1,\ldots, k_m) \in L^\infty(\mathbb{R}_+;\mathbb{R}^m)$ let us define $\mathsf{S}_t(k) \in \mathcal{U}\big(L^2(\mathbb{R}_+;\mathcal{Z})\big)$, $\mathsf{r}_t(k) \in L^2(\mathbb{R}_+;\mathcal{Z})$ and the *characteristic operator* $\widehat{\Phi}_t(k) \in \mathcal{U}(\mathcal{F})$ by

$$\begin{gathered}\big(\mathsf{S}_t(k)f\big)(s) = 1_{(0,t)}(s)\left[\mathsf{S}\big(k(s)\big) - \mathbb{1}\right] f(s) + f(s)\,, \quad \forall f \in L^2(\mathbb{R}_+;\mathcal{Z})\,,\\ \mathsf{S}\big(k(s)\big) = \prod_{\alpha=1}^m \mathrm{e}^{\mathrm{i}k_\alpha(s)B^\alpha}\,, \qquad \mathsf{r}_t(k)(s) = 1_{(0,t)}(s)\,\mathsf{r}(k;s)\,,\\ \mathsf{r}(k;s) = \mathrm{i}\sum_{\alpha=1}^m k_\alpha(s)h^\alpha(s) + \left[\mathsf{S}\big(k(s)\big) - \mathbb{1}\right] b(s)\,,\end{gathered} \tag{17}$$

$$\widehat{\Phi}_t(k) = \exp\left\{\mathrm{i}\int_0^t \mathrm{d}s\left[\sum_{\alpha=1}^m k_\alpha(s)c^\alpha(s) + \operatorname{Im}\left\langle b(s)\middle|\mathsf{S}\big(k(s)\big)b(s)\right\rangle\right]\right\} \times W\big(\mathsf{r}_t(k);\mathsf{S}_t(k)\big). \tag{18}$$

By (16) and (17), we get $\mathsf{S}(-k) = \mathsf{S}(k)^*$ and $\mathsf{S}\big(k(t)\big)^*\mathsf{r}(k;t) = -\mathsf{r}(-k;t)$.

Theorem 8 ([5, Theorem 3.1])**.** *Under Hypothesis 3, the characteristic operator introduced in Definition 6 has the following properties:*

1. localisation properties: $\widehat{\Phi}_t\big(1_{(t_1,t_2)}k\big) = \widehat{\Phi}_{t_2}\big(1_{(t_1,t_2)}k\big) \in \mathcal{U}\left(\mathcal{F}_{(t_1,t_2)}\right)$, $0 \leq t_1 < t_2 \leq t$;

2. group property: $\widehat{\Phi}_t(0) = \mathbb{1}$, $\widehat{\Phi}_t(k)\,\widehat{\Phi}_t(k') = \widehat{\Phi}_t(k+k')$, $\forall k, k' \in L^\infty(\mathbb{R}_+; \mathbb{R}^m)$*;*

3. continuity: $\widehat{\Phi}_t(\kappa k)$ *is strongly continuous in* $\kappa \in \mathbb{R}$ *and in* $t \geq 0$*;*

4. matrix elements:

$$\begin{aligned}\langle e(g)|\widehat{\Phi}_t(k)e(f)\rangle = \langle e(g)|e(f)\rangle\\ \times \exp\biggl\{\int_0^t \mathrm{d}s\biggl[-\frac{1}{2}\sum_{\alpha\beta} k_\alpha(s)\langle h^\alpha(s)|h^\beta(s)\rangle k_\beta(s)\\ + \mathrm{i}\sum_{\alpha=1}^m k_\alpha(s)\bigl(c^\alpha(s) + \langle h^\alpha(s)|f(s)\rangle + \langle g(s)|h^\alpha(s)\rangle\bigr)\\ + \bigl\langle g(s)+b(s)\bigr|\,\bigl(\mathsf{S}\bigl(k(s)\bigr)-\mathbb{1}\bigr)\,\bigl(f(s)+b(s)\bigr)\bigr\rangle\biggr]\biggr\};\end{aligned} \tag{19}$$

5. $\widehat{\Phi}_t(k)$ *is the unique unitary solution of the QSDE*

$$\mathrm{d}\widehat{\Phi}_t(k) = \sum_{i,j\geq 0} G_{ij}(t;k)\widehat{\Phi}_t(k)\,\mathrm{d}\Lambda_{ij}(t), \qquad \widehat{\Phi}_0(k) = \mathbb{1}, \tag{20a}$$

$$\begin{aligned}G_{00}(t;k) =& \langle b(t)|\,\bigl(\mathsf{S}\bigl(k(t)\bigr)-\mathbb{1}\bigr)\,b(t)\rangle\\ &+ \mathrm{i}\sum_{\alpha=1}^m k_\alpha(t)c^\alpha(t) - \frac{1}{2}\sum_{\alpha,\beta=1}^m k_\alpha(t)\langle h^\alpha(t)|h^\beta(t)\rangle k_\beta(t),\end{aligned} \tag{20b}$$

$$G_{j0}(t;k) = \langle z_j|\mathsf{r}(k;t)\rangle, \qquad G_{0j}(t;k) = -\langle \mathsf{r}(k;t)|\mathsf{S}\bigl(k(t)\bigr)z_j\rangle, \tag{20c}$$

$$G_{ij}(t;k) = \bigl\langle z_i\bigr|\,\bigl(\mathsf{S}\bigl(k(t)\bigr)-\mathbb{1}\bigr)\,z_j\bigr\rangle\,. \tag{20d}$$

Moreover, there exist a measurable space $(\Omega, \mathcal{D})$*, a projection valued measure* ξ *on* $(\Omega, \mathcal{D})$*, a family of real valued measurable functions* $\bigl\{\tilde{X}(\alpha, t; \cdot),\ \alpha = 1, \ldots, m,\ t \geq 0\bigr\}$ *on* Ω*, a family of commuting and adapted selfadjoint operators* $\bigl\{X(\alpha, t),\ \alpha = 1, \ldots, m,\ t \geq 0\bigr\}$ *such that* $\tilde{X}(\alpha, 0; \omega) = 0$*,* $X(\alpha, 0) = 0$ *and, for any choice of* n*,* $0 = t_0 < t_1 < \cdots < t_n \leq t$*,* $\kappa_\alpha^l \in \mathbb{R}$*,*

$$\begin{aligned}\widehat{\Phi}_t(k) = \exp\biggl\{\mathrm{i}\sum_{l=1}^n\sum_{\alpha=1}^m \kappa_\alpha^l\bigl[X(\alpha, t_l) - X(\alpha, t_{l-1})\bigr]\biggr\}\\ = \int_\Omega \exp\biggl\{\mathrm{i}\sum_{l=1}^n\sum_{\alpha=1}^m \kappa_\alpha^l\bigl[\tilde{X}(\alpha, t_l; \omega) - \tilde{X}(\alpha, t_{l-1}; \omega)\bigr]\biggr\}\xi(\mathrm{d}\omega)\,,\end{aligned} \tag{21}$$

where $k_\alpha(s) = \sum_{l=1}^n 1_{(t_{l-1}, t_l)}(s)\,\kappa_\alpha^l$*.*

From the unitarity and the group property we have $\widehat{\Phi}_t(k)^* = \widehat{\Phi}_t(k)^{-1} = \widehat{\Phi}_t(-k)$. Equation (20) is a right HP-equation with trivial initial space; it can be written also in left form, with the same coefficients. Note that $\overline{G_{ij}(t;-k)} = G_{ji}(t;k)$.

The observables $X(\alpha,t)$ can be identified by taking $k_\alpha(s)=\kappa$, $k_\beta(s)=0$ for $\beta\neq\alpha$; then, the first equality in (21) gives $\mathrm{e}^{\mathrm{i}\kappa X(\alpha,t)}=\widehat{\Phi}_t(k)$ and, by differentiation of the matrix elements (19), we get

$$\langle e(g)|X(\alpha,t)e(f)\rangle=\langle e(g)|e(f)\rangle\int_0^t \mathrm{d}s\big\{c^\alpha(s)\\ +\langle h^\alpha(s)|f(s)\rangle+\langle g(s)|h^\alpha(s)\rangle+\langle g(s)+b(s)|B^\alpha[f(s)+b(s)]\rangle\big\}.$$

Let us choose the complete orthonormal system $\{z_i,\ i=1,\dots,d\}$ in $\mathcal{Z}$ such that it diagonalises all the operators $B^1,\dots,B^m$ and such that its first d' components, $0\le d'\le d$, span the intersection of the null spaces of these operators; then, we have $B^\alpha=\sum_{i=d'+1}^d B_i^\alpha|z_i\rangle\langle z_i|$, $B_i^\alpha\in\mathbb{R}$, and we can write, on the exponential domain,

$$X(\alpha,t)=\int_0^t c^\alpha(s)\,\mathrm{d}s+\sum_{i=1}^{d'}\int_0^t\left(\overline{h_i^\alpha(s)}\,\mathrm{d}A_i(s)+h_i^\alpha(s)\,\mathrm{d}A_i^\dagger(s)\right)\\ +\sum_{i=d'+1}^{d}B_i^\alpha\int_0^t\left(\mathrm{d}\Lambda_{ii}+\overline{b_i(s)}\,\mathrm{d}A_i(s)+b_i(s)\,\mathrm{d}A_i^\dagger(s)+|b_i(s)|^2\,\mathrm{d}s\right). \qquad (22)$$

In quantum optical systems the continuous measurement of observables of the type $\int_0^t\left(\overline{h_i^\alpha(s)}\,\mathrm{d}A_i(s)+h_i^\alpha(s)\,\mathrm{d}A_i^\dagger(s)\right)$ can be obtained by heterodyne/homodyne detection, while terms like $\int_0^t\left(\mathrm{d}\Lambda_{ii}+\overline{b_i(s)}\,\mathrm{d}A_i(s)+b_i(s)\,\mathrm{d}A_i^\dagger(s)+|b_i(s)|^2\,\mathrm{d}s\right)$ are realised by direct detection, eventually after interference with a known signal if $b_i\neq0$ [4].

A key point in the whole construction is that even in the Heisenberg picture the observables $X(\alpha,t)$ continue to be represented by commuting operators and, so, they can be jointly measured also at different times. Let U be a right unitary cocycle representing the system-field dynamics and define $\forall T\ge0$ the "output" characteristic operator by $\widehat{\Phi}_T^{\mathrm{out}}(k):=U(T)^*\widehat{\Phi}_T(k)U(T)$. The key property giving the commutativity of the observables in the Heisenberg picture is $\widehat{\Phi}_T^{\mathrm{out}}\left(1_{(0,t)}k\right)=\widehat{\Phi}_t^{\mathrm{out}}(k)$, $0\le t\le T$. This property follows from the fact that we have $U(T)=U(T,t)U(t)$ (see Remark 1), $\widehat{\Phi}_T\left(1_{(0,t)}k\right)=\widehat{\Phi}_t(k)$ and that $U(T,t)\in\mathcal{U}\left(\mathcal{H}\otimes\mathcal{F}_{(t,T)}\right)$ commutes with $\widehat{\Phi}_t(k)\in\mathcal{U}\left(\mathcal{F}_{(0,t)}\right)$.

Let $\mathfrak{s}\in\mathcal{S}(\mathcal{H}\otimes\mathcal{F})$ be the initial system-field state. The *characteristic functional* of the process $\tilde{X}$ (the "Fourier transform" of its probability law) is given by

$$\Phi_t(k)=\mathrm{Tr}\left\{\widehat{\Phi}_t(k)U(t)\mathfrak{s}U(t)^*\right\}=\mathrm{Tr}\left\{\widehat{\Phi}_t^{\mathrm{out}}(k)\mathfrak{s}\right\}. \qquad (23)$$

All the probabilities describing the continuous measurement of the observables $X(\alpha,t)$ are contained in $\Phi_t(k)$; let us give explicitly the construction of the joint probabilities for a finite number of increments.

The measurable functions $\left\{\tilde{X}(\alpha,t;\cdot)\,,\ \alpha=1,\dots,m,\ t\ge0\right\}$, introduced in Theorem 8, represent the output signal of the continuous measurement. Let us

denote by $\Delta\tilde X(t_1,t_2) = \Big(\tilde X(1,t_2)-\tilde X(1,t_1),\ldots,\tilde X(m,t_2)-\tilde X(m,t_1)\Big)$ the vector of the increments of the output in the time interval (t_1,t_2) and by $\xi(\mathrm{d}\boldsymbol{x};t_1,t_2)$ the joint projection valued measure on $\mathbb{R}^m$ of the increments $X(\alpha,t_2)-X(\alpha,t_1)$, $\alpha=1,\ldots,m$. Note that, because of the properties of the characteristic operator, not only the different components of an increment are commuting, but also increments related to different time intervals; this implies that the projection valued measures related to different time intervals commute. Moreover, the localisation properties of the characteristic operator give

$$\xi(A;t_1,t_2)\equiv\xi\big(\Delta\tilde X(t_1,t_2)\in A\big)\in\mathcal{L}(\mathcal{F}_{(t_1,t_2)})\,,\quad \text{for any Borel set } A\subset\mathbb{R}^m. \tag{24}$$

As in the last part of Theorem 8, let us consider $0=t_0<t_1<\cdots<t_n\leq t$, $k_\alpha(s)=\sum_{l=1}^n 1_{(t_{l-1},t_l)}(s)\,\kappa_\alpha^l$; then, we can write

$$\begin{aligned}\Phi_t(k)&=\operatorname{Tr}\left\{\exp\!\left(\mathrm{i}\sum_{l=1}^n\sum_{\alpha=1}^m\kappa_\alpha^l\big[X(\alpha,t_l)-X(\alpha,t_{l-1})\big]\right)U(t)\mathfrak{s}U(t)^*\right\}\\&=\int_{\mathbb{R}^{nm}}\left(\prod_{l=1}^n\mathrm{e}^{\mathrm{i}\sum_{\alpha=1}^m\kappa_\alpha^l x_\alpha^l}\right)\mathbb{P}_{\mathfrak{s}}\big[\Delta\tilde X(t_0,t_1)\in\mathrm{d}x^1,\ldots,\Delta\tilde X(t_{n-1},t_n)\in\mathrm{d}x^n\big],\end{aligned}$$

where the physical probabilities are given by

$$\begin{aligned}&\mathbb{P}_{\mathfrak{s}}\big[\Delta\tilde X(t_0,t_1)\in A_1,\ldots,\Delta\tilde X(t_{n-1},t_n)\in A_n\big]\\&\qquad=\operatorname{Tr}\left\{\left(\prod_{l=1}^n\xi(A_j;t_{l-1},t_l)\right)U(t)\mathfrak{s}U(t)^*\right\}.\end{aligned}$$

Obviously, $\Phi_t(k)$ is the characteristic function of the physical probabilities $\mathbb{P}_{\mathfrak{s}}\big[\Delta\tilde X(t_0,t_1)\in A_1,\ldots,\Delta\tilde X(t_{n-1},t_n)\in A_n\big]$ and it uniquely determines them.

The aim is now to reformulate the continuous measurement in terms of system $S_{\mathcal{H}}$ alone, when the initial state is

$$\mathfrak{s}=\rho_0\otimes|\psi(f)\rangle\langle\psi(f)|,\qquad \rho_0\in\mathcal{S}(\mathcal{H}),\quad f\in L^2(\mathbb{R}_+;\mathcal{Z}). \tag{25}$$

Let $U(t)$ be a unitary, strongly continuous right cocycle and let us define $U(t,s)$ by Eq. (14). Let $\widehat\Phi_t(k)$ be the characteristic operator introduced in Definition 6 under Hypothesis 3 and set

$$\widehat\Phi(k;s,t):=\widehat\Phi_t\big(1_{(s,+\infty)}k\big),\qquad 0\leq s\leq t. \tag{26}$$

By the definitions (18), (26) and the points (1)-(3) of Theorem 8 one gets easily $\widehat\Phi(0;s,t)=\mathbb{1}$, $\widehat\Phi(k;r,t)=\widehat\Phi(k;r,s)\,\widehat\Phi(k;s,t)$ and that $\widehat\Phi(k;s,t)$ is strongly continuous in s and t.

Definition 7. Let us take $f\in L^2(\mathbb{R}_+;\mathcal{Z})$ and $0\leq s\leq t$. The *reduced characteristic operator* is the unique operator $\mathcal{G}_f(k;s,t):\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H})$ that satisfies, $\forall u,v\in\mathcal{H}$, $\forall X\in\mathcal{L}(\mathcal{H})$,

$$\big\langle v\big|\mathcal{G}_f(k;s,t)[X]u\big\rangle=\big\langle U(t,s)v\otimes\psi(f)\big|\big(X\otimes\widehat\Phi(k;s,t)\big)U(t,s)u\otimes\psi(f)\big\rangle. \tag{27}$$

Then, $\mathcal{T}_f(s,t) := \mathcal{G}_f(0;s,t)$ represents the *reduced evolution operator* for the observables of $S_{\mathcal{H}}$.

Theorem 9. *In the hypotheses above, the family of linear maps $\mathcal{G}_f(k;s,t)$, $t \geq s \geq 0$, $f \in L^2(\mathbb{R}_+;\mathcal{Z})$, $k \in L^\infty(\mathbb{R}_+;\mathbb{R}^d)$, has the following properties:*

1. $\mathcal{G}_f(k;s,s) = \mathbb{1}$; $\qquad \|\mathcal{G}_f(k;s,t)\| \leq 1$;
2. $\mathcal{G}_f(k;s,t)$ *is* completely positive definite, *i.e., for all integers n, test functions k^i, vectors ϕ_i and operators X_i, one has*
$$\sum_{i,j=1}^{n} \left\langle \phi_i \middle| \mathcal{G}_f(k^i - k^j;s,t)[X_i^* X_j]\phi_j \right\rangle \geq 0\,;$$
3. $\mathcal{G}_f(k;s,t)$ *is a σ-weakly continuous operator on $\mathcal{L}(\mathcal{H})$ and it has a pre-adjoint $\mathcal{G}_f(k;s,t)_*$ acting on the trace class on $\mathcal{H}$;*
4. *for each $X \in \mathcal{L}(\mathcal{H})$ the maps $t \mapsto \mathcal{G}_f(k;s,t)[X]$, $s \mapsto \mathcal{G}_f(k;s,t)[X]$ and $\kappa \mapsto \mathcal{G}_f(\kappa k;s,t)[X]$ are continuous with respect to the σ-weak topology of $\mathcal{L}(\mathcal{H})$;*
5. $\forall u,v \in \mathcal{H}$, $\forall X \in \mathcal{L}(\mathcal{H})$,
$$\left\langle v \middle| \mathcal{G}_f(k;s,t)[X]u \right\rangle = \left\langle U(t,s)v \otimes \psi(f_{(s,t)}) \middle| \big(X \otimes \widehat{\Phi}(k;s,t)\big) U(t,s)u \otimes \psi(f_{(s,t)}) \right\rangle;$$
6. *if $f(x) = g(x)$ for all $x \in (s,t)$, we get $\mathcal{G}_f(k;s,t) = \mathcal{G}_g(k;s,t)$; then, $\mathcal{G}_f(k;s,t)$ is well defined for all $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathcal{Z})$;*
7. $\mathcal{G}_f(k;r,s) \circ \mathcal{G}_f(k;s,t) = \mathcal{G}_f(k;r,t), \quad 0 \leq r \leq s \leq t$;
8. *for all $s,t \geq 0$ we have $\mathcal{G}_f(k;s,s+t) = \mathcal{G}_{f_s}(k_s;0,t)\Big|_{h\to h_s,\, b\to b_s,\, c\to c_s}$, where we have introduced the shifted functions $f_s(x) = f(x+s)$, $k_s(x) = k(x+s)$, $h_s(x) = h(x+s)$, $b_s(x) = b(x+s)$, $c_s(x) = c(x+s)$.*

Moreover, the evolution operator $\mathcal{T}_f(s,t)$, $t \geq s \geq 0$, $f \in L^2(\mathbb{R}_+;\mathcal{Z})$, introduced in Definition 7, enjoys the properties:

(i) $\mathcal{T}_f(s,t)[\mathbb{1}] = \mathbb{1}$; $\qquad \mathcal{T}_f(s,s) = \mathbb{1}$; $\qquad \|\mathcal{T}_f(s,t)\| = 1$;

(ii) *$\mathcal{T}_f(s,t)$ is a σ-weakly continuous operator on $\mathcal{L}(\mathcal{H})$ and it has a pre-adjoint $\mathcal{T}_f(s,t)_*$ acting on the trace class on $\mathcal{H}$;*

(iii) *$\mathcal{T}_f(s,t)$ is completely positive;*

(iv) *for each $X \in \mathcal{L}(\mathcal{H})$ the maps $t \mapsto \mathcal{T}_f(s,t)[X]$ and $s \mapsto \mathcal{T}_f(s,t)[X]$ are continuous with respect to the σ-weak topology of $\mathcal{L}(\mathcal{H})$;*

(v) $\mathcal{T}_f(r,s) \circ \mathcal{T}_f(s,t) = \mathcal{T}_f(r,t), \quad 0 \leq r \leq s \leq t$;

(vi) *if $f(x) = g(x)$ for all $x \in (s,t)$, we have $\mathcal{T}_f(s,t) = \mathcal{T}_g(s,t)$; then, $\mathcal{T}_f(s,t)$ is well defined for all $f \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{Z})$;*

(vii) *for all $s,t \geq 0$ we have $\mathcal{T}_f(s,s+t) = \mathcal{T}_{f_s}(0,t)$, where $f_s(x) = f(x+s)$.*

Proof. The first statement of point (1) is immediate from the fact that $\widehat{\Phi}(k;s,s) = \mathbb{1}$. The second statement follows from $\|\mathcal{G}_f(k;s,t)[X]\| \leq \left\| X \otimes \widehat{\Phi}(k;s,t) \right\| = \|X\|$; the first step is from the definition (27), the unitarity of $U(t,s)$ and the normalisation of the coherent vector $\psi(f)$, while the second step is due to the unitarity of the characteristic operator.

By using $\widehat{\Phi}(k^i - k^j;t,s) = \widehat{\Phi}(-k^i;t,s)^* \ \widehat{\Phi}(-k^j;t,s)$ and the definition of $\mathcal{G}_f(k;s,t)$, one gets immediately

$$\sum_{i,j=1}^n \left\langle \phi_i \middle| \mathcal{G}_f(k^i - k^j;s,t)[X_i^* X_j] \phi_j \right\rangle \\ = \left\| \sum_{j=1}^n X_j \otimes \widehat{\Phi}(-k^j;t,s)\, U(t,s)\, \phi_j \otimes \psi(f) \right\|^2 \geq 0,$$

which is point (2).

Any $\tau \in \mathcal{T}(\mathcal{H})$ can be written as $\tau = \sum_n |u_n\rangle\langle v_n|$ for some choice of the vectors u_n, v_n in $\mathcal{H}$. Then, we have

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}} \left\{ \mathcal{G}_f(k;s,t)[X]\tau \right\} &= \sum_n \langle v_n | \mathcal{G}_f(k;s,t)[X] u_n \rangle \\ &= \sum_n \left\langle U(t,s)\, v_n \otimes \psi(f) \middle| \left(X \otimes \widehat{\Phi}(k;s,t) \right) U(t,s)\, u_n \otimes \psi(f) \right\rangle \\ &= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{F}} \left\{ \left(X \otimes \widehat{\Phi}(k;s,t) \right) U(t,s)\, (\tau \otimes |\psi(f)\rangle\langle\psi(f)|)\, U(t,s)^* \right\} \\ &=: \mathrm{Tr}_{\mathcal{H}} \left\{ X \mathcal{G}_f(k;s,t)_*][\tau] \right\}, \end{aligned}$$

which defines the pre-adjoint. The existence of the pre-adjoint of $\mathcal{G}_f(k;s,t)$ implies its σ-weak continuity [24, Corollary of Theorem 1.13.2, p. 29] and this completes the proof of point (3)

By point (1) $\mathcal{G}_f(\kappa k;s,t)$ is bounded uniformly in s, t and κ. By Proposition 1.15.2 in [24], the weak and the σ-weak topologies are equivalent on the bounded spheres; so, it is enough to prove the weak continuity. Let us set $\phi_1 := U(t,s)\, v \otimes$

$\psi(f)$, $\phi_2 := U(t,s)\,u\otimes\psi(f)$, $\tilde X := X\otimes\widehat\Phi(k;s,t)$. Then, we have

$$
\begin{aligned}
&|\langle v|\mathcal{G}_f(k;s,t+\epsilon)[X]u\rangle - \langle v|\mathcal{G}_f(k;s,t)[X]u\rangle| \\
&\qquad = \left|\langle U(t+\epsilon,t)\phi_1|\tilde X\widehat\Phi(k;t,t+\epsilon)U(t+\epsilon,t)\phi_2\rangle - \langle\phi_1|\tilde X\phi_2\rangle\right| \\
&\qquad\quad \le \left|\langle\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_1|\tilde X\widehat\Phi(k;t,t+\epsilon)U(t+\epsilon,t)\phi_2\rangle\right| \\
&+\left|\langle\phi_1|\tilde X\widehat\Phi(k;t,t+\epsilon)\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_2\rangle\right| + \left|\langle\phi_1|\tilde X\left(\widehat\Phi(k;t,t+\epsilon)-\mathbb{1}\right)\phi_2\rangle\right| \\
&\qquad \le \left\|\tilde X\widehat\Phi(k;t,t+\epsilon)U(t+\epsilon,t)\right\|\|\phi_2\|\left\|\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_1\right\| \\
&\qquad\quad + \left\|\tilde X\widehat\Phi(k;t,t+\epsilon)\right\|\|\phi_1\|\left\|\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_2\right\| \\
&\quad + \left\|\tilde X\right\|\|\phi_1\|\left\|\left(\widehat\Phi(k;t,t+\epsilon)-\mathbb{1}\right)\phi_2\right\| \le \|X\|\Big\{\|u\|\left\|\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_1\right\| \\
&\qquad\quad + \|v\|\left[\left\|\left(U(t+\epsilon,t)-\mathbb{1}\right)\phi_2\right\| + \left\|\left(\widehat\Phi(k;t,t+\epsilon)-\mathbb{1}\right)\phi_2\right\|\right]\Big\},
\end{aligned}
$$

which gives the continuity in t. The continuity in s can be proved in a similar way. By similar steps we get

$$
\begin{aligned}
&|\langle v|\mathcal{G}_f(\kappa' k;s,t)[X]u\rangle - \langle v|\mathcal{G}_f(\kappa k;s,t)[X]u\rangle| \\
&\qquad\qquad \le \|X\|\,\|v\|\left\|\left(\widehat\Phi(\kappa' k;s,t)-\widehat\Phi(\kappa k;s,t)\right)U(t,s)u\otimes\psi(f)\right\|,
\end{aligned}
$$

which gives the continuity in κ, due to point (3) in Theorem 8. This ends the proof of point (4). Points (5) and (6) are immediate by the localisation properties.

By using the identification $\psi(f)=\psi(f_{(0,r)})\otimes\psi(f_{(r,s)})\otimes\psi(f_{(s,t)})\otimes\psi(f_{(t}})$ and the localisation properties $\widehat\Phi(k;a,b)\in\mathcal{U}(\mathcal{F}_{(a,b)})$, $U(b,a)\in\mathcal{U}(\mathcal{H}\otimes\mathcal{F}_{(a,b)})$, we have

$$
\begin{aligned}
&\langle v|\mathcal{G}_f(k;r,s)\circ\mathcal{G}_f(k;s,t)[X]u\rangle \\
&= \Big\langle U(s,r)\left(v\otimes\psi(f_{(r,s)})\right) \,\Big|\, \left(\mathcal{G}_f(k;s,t)[X]\otimes\widehat\Phi(k;r,s)\right)U(s,r)\left(u\otimes\psi(f_{(r,s)})\right)\Big\rangle \\
&\qquad = \langle U(t,s)\left[U(s,r)\left(v\otimes\psi(f_{(r,s)})\right)\otimes\psi(f_{(s,t)})\right] \,| \\
&\left(\left(X\otimes\widehat\Phi(k;s,t)\right)\otimes\widehat\Phi(k;r,s)\right)U(t,s)\left[U(s,r)\left(u\otimes\psi(f_{(r,s)})\right)\otimes\psi(f_{(s,t)})\right]\rangle \\
&\qquad = \Big\langle U(t,r)\left(v\otimes\psi(f_{(r,t)})\right)\,\Big|\,\left(X\otimes\widehat\Phi(k;r,t)\right)U(t,r)\left(u\otimes\psi(f_{(r,t)})\right)\Big\rangle \\
&\qquad\qquad = \langle v|\mathcal{G}_f(k;r,t)[X]u\rangle,
\end{aligned}
$$

which gives point (7). Finally, by Eqs. (5), (14), (15), (19) we have

$$
\begin{aligned}
&\big\langle v\big|\mathcal{G}_f(k;s,s+t)[X]u\big\rangle \\
&\qquad = \big\langle U(t)v\otimes\psi(f_s)\big|\big(X\otimes\Theta_s\widehat\Phi(k;s,s+t)\Theta_s^*\big)U(t)u\otimes\psi(f_s)\big\rangle \\
&\quad = \big\langle U(t,0)v\otimes\psi(f_s)\big|\left(X\otimes\widehat\Phi(k_s;0,t)\Big|_{h\to h_s,\,b\to b_s,\,c\to c_s}\right)U(t,0)u\otimes\psi(f_s)\big\rangle,
\end{aligned}
$$

and point (8) follows.

By the particularising the previous statements to the case $k = 0$, we get the properties of the evolution operator. □

The definition of the reduced characteristic operator has been given in such a way that it is sufficient to construct the characteristic functional (23) when the initial state is given by Eq. (25): $\Phi_t(k) = \mathrm{Tr}\left\{\mathcal{G}_f(k;0,t)[\mathbb{1}]\rho_0\right\}$. So, the reduced characteristic operator determines all the probabilities of the output. However, the reduced characteristic operator gives something more: the states after the measurement, conditional on the observed output. This is obtained through the correspondence with the *instruments* representing the continuous measurement, see [5, pp. 244–245] and [2].

5 The evolution equations

Up to now, we have only made use of the cocycle properties of $U(t)$, but we are interested in finding the infinitesimal generator and the evolution equation of the reduced characteristic operator and for that we need also the QSDE for $U(t)$. The reduced characteristic operator comes out from the product of three terms: the operators $\widehat{\Phi}_t(k)$, $U(t)$ and $U(t)^*$. To compute the differential of this product we have to use two times the second fundamental formula of quantum stochastic calculus.

Our first step will be to differentiate the unitary process

$$\Psi_t(k) := \left(\mathbb{1} \otimes \widehat{\Phi}_t(k)\right) U(t), \qquad k \in L^\infty(\mathbb{R}_+;\mathbb{R}^m); \tag{28}$$

then, we shall use the second fundamental formula of quantum stochastic calculus to elaborate the expression giving the reduced characteristic operator.

Lemma 10. *Let Hypotheses 1, 2, 3 hold and the functions $c(t)$, $b(t)$, $h^\alpha(t)$ be locally bounded in time. Then, $\Psi_t(k)$, defined by (28), can be expressed as the quantum stochastic integral on $\mathrm{Dom}(C^{1/2}) \odot \mathcal{E}$*

$$\Psi_t(k) = \mathbb{1} + \sum_{i,j\geq 0} \int_0^t \left(\mathbb{1} \otimes \widehat{\Phi}_s(k)\right) M_{ij}(s;k) U(s) \mathrm{d}\Lambda_{ij}(s), \tag{29}$$

where

$$M_{00}(t;k) = K + \sum_{r=1}^{d} \langle \mathsf{r}(-k;t)|z_r\rangle R_r + \Big\{ \langle b(t)| \left(\mathsf{S}\big(k(t)\big) - \mathbb{1}\right) b(t)\rangle + \mathrm{i} \sum_{\alpha=1}^{m} k_\alpha(t) c^\alpha(t) - \frac{1}{2} \sum_{\alpha,\beta=1}^{m} k_\alpha(t) \langle h^\alpha(t)|h^\beta(t)\rangle k_\beta(t) \Big\} \mathbb{1}, \tag{30a}$$

$$M_{0j}(t;k) = N_j + \sum_{r=1}^{d} \langle \mathsf{r}(-k;t)|z_r\rangle S_{rj}\,, \qquad j \geq 1\,, \tag{30b}$$

$$M_{i0}(t;k) = \sum_{r\geq 1} \big\langle z_i \big| \mathsf{S}\big(k(t)\big) z_r \big\rangle\, R_r + \langle z_i | \mathsf{r}(k;t)\rangle \mathbb{1}, \qquad i \geq 1\,, \tag{30c}$$

$$M_{ij}(t;k) = \sum_{r\geq 1} \big\langle z_i \big| \mathsf{S}\big(k(t)\big) z_r \big\rangle\, S_{rj} - \delta_{ij}\mathbb{1}, \qquad i,j \geq 1, \tag{30d}$$

with $\mathrm{Dom}\big(M_{00}(t;k)\big) = \mathrm{Dom}(K)$, $\mathrm{Dom}\big(M_{0j}(t;k)\big) = \mathrm{Dom}(N_j) \supset \mathrm{Dom}(K)$, $\mathrm{Dom}\big(M_{i0}(t;k)\big) \supset \bigcap_{k=1}^{d} \mathrm{Dom}(R_k) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*)$, $\mathrm{Dom}\big(M_{ij}(t;k)\big) = \mathcal{H}$, $i,j = 1,\dots,d$.

Moreover, $\forall f,g \in \mathcal{M}$ *and* $\forall u,v \in \mathrm{Dom}(C^{1/2})$*, one has*

$$\begin{aligned}
&\langle U(t)v\otimes e(g)| \left(X\otimes \widehat{\Phi}_t(k)\right) U(t)u\otimes e(f)\rangle = \langle v|Xu\rangle\langle e(g)|e(f)\rangle \\
&\quad + \sum_{i,j\geq 0}\int_0^t \mathrm{d}s\, \overline{g_i(s)} \Big\{ \big\langle U(s)v\otimes e(g)\big| \big(X\otimes\widehat{\Phi}_s(k)\big) M_{ij}(s;k) U(s)u\otimes e(f)\big\rangle \\
&\qquad\qquad + \langle F_{ji}U(s)v\otimes e(g)| \big(X\otimes\widehat{\Phi}_s(k)\big) U(s)u\otimes e(f)\rangle \\
&\quad\ + \sum_{l\geq 1} \big\langle F_{li}U(s)v\otimes e(g)\big| \big(X\otimes\widehat{\Phi}_s(k)\big) M_{lj}(s;k) U(s)u\otimes e(f)\big\rangle \Big\} f_j(s).
\end{aligned} \tag{31}$$

Let us recall the convention $f_0(s) = g_0(s) = 1$.

Proof. By Eqs. (8) and (20), the second fundamental formula of quantum stochastic calculus, $\Phi_t(-k) = \Phi_t(k)^*$ and $\overline{G_{ji}(s;-k)} = G_{ij}(t;k)$, we get for $f,g \in \mathcal{M}$ and $u,v \in \mathrm{Dom}(C^{1/2})$

$$\begin{aligned}
&\big\langle v\otimes e(g)\big| \Psi_t(k) u\otimes e(f)\big\rangle - \langle v|u\rangle\langle e(g)|e(f)\rangle \\
&\qquad = \sum_{i,j\geq 0}\int_0^t \mathrm{d}s\, \overline{g_i(s)} \big\langle v\otimes \widehat{\Phi}_t(-k)e(g)\big| M_{ij}(s;k)U(s)u\otimes e(f)\rangle f_j(s),
\end{aligned} \tag{32}$$

where

$$M_{ij}(s;k) := F_{ij} + G_{ij}(s;k)\mathbb{1} + \sum_{r\geq 1} G_{ir}(s;k) F_{rj}\,. \tag{33}$$

By inserting the explicit expressions of the elements of the matrices F and G into Eq. (33) we get Eqs. (30). The statements about the domains follow from Hypothesis 1 point (iv), Proposition 1 point (1) and the fact that the operators S_{ij} are bounded.

It is easy to check that the processes $\big(\mathbb{1}\otimes\widehat{\Phi}_s(k)\big) M_{ij}(s;k)U(s)$ are stochastically integrable, by using the fact that $\widehat{\Phi}_s(k)$ is unitary, the functions $G_{ij}(s;k)$ are locally bounded, due to the boundedness assumption on c, b, h^α, and the processes $F_{ij}U(s)$ are stochastically integrable by hypothesis. Then, Eq. (29) follows from Eq. (32) and the first fundamental formula of quantum stochastic calculus.

By the second fundamental formula of quantum stochastic calculus applied to $(X^*\otimes\mathbb{1})U(t)$ and $\Psi_t(k)$ we get immediately Eq. (31). $\square$

For $\lambda, \mathsf{r} \in \mathcal{Z}$ (with components denoted by λ_j and r_j) let us define the operators

$$B_i(\lambda) := R_i + \sum_{j=1}^{d} S_{ij}\lambda_j, \qquad i = 1, \dots, d, \tag{34a}$$

$$K(\lambda, \mathsf{r}) := K - \sum_{i,j=1}^{d} R_i^* S_{ij}\lambda_j - \frac{\|\lambda\|^2}{2}\,\mathbb{1} + \sum_{i=1}^{d} \overline{\mathsf{r}_i}\, B_i(\lambda). \tag{34b}$$

By taking into account Hypothesis 1 and Proposition 1 we have

$$\mathrm{Dom}\big(B_i(\lambda)\big) = \mathrm{Dom}(R_i) \supset \mathrm{Dom}(K) \cup \mathrm{Dom}(K^*),$$
$$\mathrm{Dom}\big(K(\lambda, \mathsf{r})\big) = \mathrm{Dom}(K) \supset \mathrm{Dom}(C^{1/2}).$$

Again by Hypothesis 1 and Proposition 1, the domains of the adjoint of the previous operators contain $\mathrm{Dom}(F^*) \supset \widetilde{D}$ and on $\mathrm{Dom}(F^*)$ we have

$$B_i(\lambda)^* = R_i^* + \sum_{j=1}^{d} \overline{\lambda_j}\, S_{ij}^*, \qquad i = 1, \dots, d, \tag{35a}$$

$$K(\lambda, \mathsf{r})^* = K^* - \sum_{i,j=1}^{d} \overline{\lambda_j}\, S_{ij}^* R_i - \frac{\|\lambda\|^2}{2}\,\mathbb{1} + \sum_{i=1}^{d} \mathsf{r}_i\, B_i(\lambda)^*. \tag{35b}$$

Finally, for κ, $c \in \mathbb{R}^m$, $b \in \mathcal{Z}$, $h \in \mathcal{Z}^m$ we define also

$$C(\kappa, b, c, h) := \langle b|\big(\mathsf{S}(\kappa) - \mathbb{1}\big)b\rangle + \mathrm{i}\sum_{\alpha=1}^{m} \kappa_\alpha c^\alpha - \frac{1}{2}\sum_{\alpha,\beta=1}^{m} \kappa_\alpha \langle h^\alpha | h^\beta\rangle \kappa_\beta\,. \tag{36}$$

Proposition 11. *Let Hypotheses 1, 2, 3 hold and the functions $c(t)$, $b(t)$, $h^\alpha(t)$ be locally bounded in time. Then, $\forall f \in \mathcal{M}$, $\forall k \in L^\infty(\mathbb{R}_+; \mathbb{R}^m)$, $\forall u, v \in \mathrm{Dom}(C^{1/2})$ we have*

$$\begin{aligned}
&\langle U(t)v \otimes \psi(f)|\, \Big(X \otimes \widehat{\Phi}_t(k)\Big)\, U(t)u \otimes \psi(f)\rangle = \langle v|Xu\rangle \\
&\quad + \int_0^t \mathrm{d}s\, \Big\{ \big\langle U(s)v \otimes \psi(f)\big|\big(X \otimes \widehat{\Phi}_s(k)\big) K\big(f(s), \mathsf{r}(-k,s)\big) U(s)u \otimes \psi(f)\big\rangle \\
&\qquad + \langle K\big(f(s), \mathsf{r}(k,s)\big) U(s)v \otimes \psi(f)|\big(X \otimes \widehat{\Phi}_s(k)\big) U(s)u \otimes \psi(f)\rangle \\
&+ \sum_{i,j=1}^{d} \langle z_i|\mathsf{S}\big(k(s)\big)z_j\rangle \big\langle B_i\big(f(s)\big) U(s)v \otimes \psi(f)\big|\big(X \otimes \widehat{\Phi}_s(k)\big) B_j\big(f(s)\big) U(s)u \otimes \psi(f)\big\rangle \\
&\quad + C\big(k(s), b(s), c(s), h(s)\big) \big\langle U(s)v \otimes \psi(f)\big|\big(X \otimes \widehat{\Phi}_s(k)\big) U(s)u \otimes \psi(f)\big\rangle \Big\}.
\end{aligned} \tag{37}$$

Proof. The statement follows by direct computations, by inserting the explicit expressions of F_{ij}, $M_{ij}(t;k)$, N_j into Eq. (31) with $g = f$. □

Let $\mathcal{D} \subset \mathcal{L}(\mathcal{H})$ be the linear span of the rank-one operators of the type $|\psi\rangle\langle\phi|$ with $\psi, \phi \in \mathrm{Dom}(F^*)$. By using the operators (35), we define, $\forall\psi, \phi \in \mathrm{Dom}(F^*)$, $\forall u, v \in \mathcal{H}$,

$$\langle v|\mathcal{K}_f^k(t)[|\psi\rangle\langle\phi|]u\rangle = \langle v|\psi\rangle\langle K\big(f(t), \mathsf{r}(-k,t)\big)^*\phi|u\rangle + \langle v|K\big(f(t), \mathsf{r}(k,t)\big)^*\psi\rangle\langle\phi|u\rangle + \sum_{i,j=1}^{d}\langle z_i|\mathsf{S}\big(k(t)\big)z_j\rangle\langle v|B_i\big(f(t)\big)^*\psi\rangle\langle B_j\big(f(t)\big)^*\phi|u\rangle + C\big(k(t), b(t), c(t), h(t)\big)\langle v|\psi\rangle\langle\phi|u\rangle; \quad (38)$$

then, by linearity, we extend $\mathcal{K}_f^k(t)$ to $\mathcal{D}$.

Corollary 12. *Let Hypotheses 1, 2, 3 hold and the functions $c(t)$, $b(t)$, $h^\alpha(t)$ be locally bounded in time. Then, $\forall f \in \mathcal{M}$, $\forall k \in L^\infty(\mathbb{R}_+;\mathbb{R}^m)$, $\forall u, v \in \mathcal{H}$, $\forall X \in \mathcal{D}$, we have*

$$\langle v\big|\mathcal{G}_f(k;0,t)[X]u\rangle = \langle v|Xu\rangle + \int_0^t \langle v\big|\mathcal{G}_f(k;0,s)\big[\mathcal{K}_f^k(s)[X]\big]u\rangle \mathrm{d}s\,. \quad (39)$$

Proof. By using the notations above, Proposition 11 gives immediately Eq. (39) $\forall u, v \in \mathrm{Dom}(C^{1/2})$. Being $X \in \mathcal{D}$, the operator $\mathcal{K}_f^k(s)[X]]$ turns out to be bounded; moreover, we have $\|\mathcal{G}_f(k;s,t)\| \leq 1$. Then, we can extend (39) to any $u, v \in \mathcal{H}$. □

By introducing the pre-adjoint of $\mathcal{G}_f(k;0,t)$ and extending (39) to the whole trace class we get: $\forall X \in \mathcal{D}$, $\forall\tau \in \mathcal{T}(\mathcal{H})$,

$$\mathrm{Tr}_{\mathcal{H}}\left\{X\mathcal{G}_f(k;0,t)_*[\tau]\right\} = \mathrm{Tr}_{\mathcal{H}}\left\{X\tau\right\} + \int_0^t \mathrm{Tr}_{\mathcal{H}}\left\{\mathcal{K}_f^k(s)[X]\mathcal{G}_f(k;0,s)_*[\tau]\right\}\mathrm{d}s\,, \quad (40)$$

with initial condition $\mathcal{G}_f(k;0,0)_* = \mathbb{1}$. For $k = 0$ and $\tau \in \mathcal{S}(\mathcal{H})$, Eq. (40) is a *quantum master equation* and the formal pre-adjoint of $\mathcal{K}_f^0(t)$ is known as *Liouville operator*. We can say that (40) is a generalisation of a quantum master equation, which includes the continuous measurement.

The problem which remains open is to prove the uniqueness of the solution of Eq. (39) or of Eq. (40). We note that in the case of quantum dynamical semigroups the positivity plays a role in the analogous problem, see Theorem 4, while in the case of Eq. (39) we have only that $\mathcal{G}$ is positive definite in k.

6 An example: the degenerate parametric oscillator

The degenerate parametric oscillator is the physical system which was used to produce squeezed light. [19,26,27] The squeezing of the light was revealed by balanced homodyne detection, a measurement scheme which is indeed described by continuous measurements of diffusive type. [4,6]. Such a quantum optical

system is constituted by an optical cavity closed by two partially transparent mirrors with inside a crystal with a $\chi^{(2)}$ non-linearity. Only two cavity modes of the electromagnetic field inside the cavity are relevant: the subharmonic field of frequency ω_C (a quantum oscillator with annihilation and creation operators a, $a^\dagger$) and the pump field of double frequency (with annihilation and creation operators b, $b^\dagger$). The pump field is populated by a resonant laser entering the cavity, the crystal couples the two modes and the light coming out of the cavity is detected by homodyne devices and/or photocounters. The degenerate parametric oscillator is well studied from the point of view of theoretical physics and quantum optics in [11, Chapts. 9, 10, 12, 18]. Here we want to prove that the mathematical model of the degenerate parametric oscillator with direct and homodyne detection can be rigourously formulated and gives an example of the theory we have developed.

The formal master equation is given in [11, Eq. (9.97)] and reads

$$\begin{multline}\dot\rho(t) = -\mathrm{i}[H_0, \rho(t)] - \mathrm{i}\left[\lambda \mathrm{e}^{-2\mathrm{i}\omega_C t} b^\dagger + \overline{\lambda}\mathrm{e}^{2\mathrm{i}\omega_C t} b,\, \rho(t)\right] \\ + \kappa\left(\overline{n}+1\right)\left(2a\rho(t)a^\dagger - a^\dagger a\rho(t) - \rho(t)a^\dagger a\right) \\ + \kappa\overline{n}\left(2a^\dagger\rho(t)a - aa^\dagger\rho(t) - \rho(t)aa^\dagger\right) + \kappa_p\overline{n}_p\left(2b^\dagger\rho(t)b - bb^\dagger\rho(t) - \rho(t)bb^\dagger\right) \\ + \kappa_p\left(\overline{n}_p+1\right)\left(2b\rho(t)b^\dagger - b^\dagger b\rho(t) - \rho(t)b^\dagger b\right). \quad (41)\end{multline}$$

The Hamiltonian term H_0 contains the free energies of the modes and the interaction due to the $\chi^{(2)}$ non-linearity:

$$H_0 = \omega_C a^\dagger a + 2\omega_C b^\dagger b + \frac{\mathrm{i}g}{2}\left(a^{\dagger 2}b - b^\dagger a^2\right). \quad (42)$$

For the constants we have $\omega_C > 0$, $g \in \mathbb{R}$, $g \neq 0$, $\kappa > 0$, $\overline{n} \geq 0$, $\kappa_p > 0$, $\overline{n}_p \geq 0$.

This model, plus detection, can be rigourously formulated in the set up developed before. First, the Hilbert space is identified with the span of the eigenvectors of the two number operators and the creation and annihilation operators are defined. Let us take the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ with canonical orthonormal basis $\{e_{n,m},\ n,m \geq 0\}$. The creation, annihilation and number operators for the subharmonic mode are defined by

$$\mathrm{Dom}(a) = \mathrm{Dom}\left(a^\dagger\right) = \left\{ u \in \mathcal{H} : \sum_{n,m\geq 0} n\left|u_{n,m}\right|^2 < +\infty \right\}, \quad (43a)$$

$$a^\dagger e_{n,m} = \sqrt{n+1}\, e_{n+1,m}, \qquad a\, e_{0,m} = 0, \quad a\, e_{n,m} = \sqrt{n}\, e_{n-1,m}, \text{ if } n > 0, \quad (43b)$$

$$\mathrm{Dom}\left(a^\dagger a\right) = \left\{ u \in \mathcal{H} : \sum_{n,m\geq 0} n^2\left|u_{n,m}\right|^2 < +\infty \right\}, \qquad a^\dagger a\, e_{n,m} = n\, e_{n,m}. \quad (43c)$$

An analogous definition holds for the operators $b^\dagger$, b, $b^\dagger b$, which act on the second factor of the tensor product.

In constructing the model we have to reproduce the effects contained in the master equation (41) and to introduce the measurement. So, we have to introduce losses at the mirrors and thermal dissipation in the crystal and at the walls of the cavity. We have also to introduce the possibility of injecting laser light feeding the pump mode. Moreover, we consider the direct detection of photons with two photocounters, chosen one to be sensible to photons around frequency ω_C and the other to photons around frequency $2\omega_C$. Finally, we consider homodyning around frequency ω_C. To realise all these features in the mathematical model we need many channels, but some channels with similar structure can be collected together and the minimal number is $d = 8$. We use channels 1 and 2 to describe the light reaching the two photocounters and channel 3 for the light reaching the homodyne detector, channel 4 is the one used for the injection of the laser, channels 5–8 describe losses and thermal dissipation. There is no scattering effect which mixes the channels. The channel operators and the unitary matrix of system operators we need are

$$R_1 = \beta_1 \, b, \qquad R_2 = \alpha_1 \, a, \qquad R_3 = \alpha_2 \, a, \qquad R_4 = \beta_2 \, b, \qquad R_5 = \beta_3 \, b, \tag{44a}$$

$$R_6 = \alpha_3 \, a, \qquad R_7 = \beta_4 \, b^\dagger, \qquad R_8 = \alpha_4 \, a^\dagger, \qquad S_{ij} = \delta_{ij} \mathbb{1}, \tag{44b}$$

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 2\kappa \left(\overline{n} + 1 \right), \qquad |\alpha_4|^2 = 2\kappa\overline{n}, \tag{44c}$$

$$|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 = 2\kappa_p \left(\overline{n}_p + 1 \right), \qquad |\beta_4|^2 = 2\kappa_p\overline{n}_p. \tag{44d}$$

The operator K has to include the Hamiltonian H_0 and to satisfy Eq. (10); so, it must have the formal expression

$$\begin{aligned} K = -\mathrm{i}H_0 - \frac{1}{2} \sum_{i=1}^{8} {R_i}^* R_i = \frac{g}{2} \left(a^{\dagger 2} b - b^\dagger a^2 \right) - \left(\kappa\overline{n} + \kappa_p \overline{n}_p \right) \mathbb{1} \\ - \left(\mathrm{i}\omega_C + \kappa \left(2\overline{n} + 1 \right) \right) a^\dagger a - \left(2\mathrm{i}\omega_C + \kappa_p \left(2\overline{n}_p + 1 \right) \right) b^\dagger b. \end{aligned}$$

Rigourously, by defining $u_{n,m} = 0$ if $n < 0$ and/or $m < 0$, we have

$$Ku = \sum_{n,m} k_u(n,m) \, e_{n,m} \,, \qquad \mathrm{Dom}(K) = \left\{ u : \sum_{n,m} \left| k_u(n,m) \right|^2 < +\infty \right\}, \tag{45a}$$

$$\begin{aligned} k_u(n,m) := \frac{g}{2} \sqrt{n(n-1)(m+1)} \, u_{n-2,m+1} - \frac{g}{2} \sqrt{m(n+1)(n+2)} \, u_{n+2,m-1} \\ - \left[\kappa\overline{n} + \kappa_p\overline{n}_p + \mathrm{i}\omega_C n + \kappa \left(2\overline{n} + 1 \right) n + 2\mathrm{i}\omega_C m + \kappa_p \left(2\overline{n}_p + 1 \right) m \right] u_{n,m} \,. \end{aligned} \tag{45b}$$

Theorem 13. *Let us construct the F-matrix by setting $F_{00} = K$, $F_{i0} = R_i$, $F_{0j} = N_j =: -R_j^*$, $F_{ij} = 0$, $i, j \geq 1$. Then, Hypothesis 1 hold true with $D = \widetilde{D}$ given by the linear span of the basis $\{e_{n,m},\ n, m \geq 0\}$ and with $C = N^4$, where $N := a^\dagger a + 2b^\dagger b$.*

Proof. By applying the definition of adjoint and Riesz lemma [22] one can easily check that $a^* = a^\dagger$, $a^{\dagger\,*} = a$ and the same for b, $b^\dagger$, as it is well known. In

particular all operators R_i, N_i are closed [14]. By (43a) we have the domain

$$D_{RN} := \bigcap_{\substack{i,\, j \\ (i,j)\neq(0,0)}} \mathrm{Dom}\,(F_{ij}) = \left\{ u \in \mathcal{H} : \sum_{n,m\geq 0} (n+m)\, |u_{n,m}|^2 < +\infty \right\}. \tag{46}$$

Again by the definition of adjoint, [22] we get K^*, which turns out to be defined by Eqs. (45) with the substitutions $\omega_C \to -\omega_C$, $g \to -g$. From the definitions of K and K^* we get, by direct computations, the dissipativity conditions (10) and $K^{**} = K$. In particular also K is closed.

For every $u \in D$ we get, from the dissipativity condition

$$\|Ku\|^2 \geq \left\langle Ku \Big| \frac{u}{\|u\|} \right\rangle \left\langle \frac{u}{\|u\|} \Big| Ku \right\rangle = \frac{|\langle Ku|u\rangle|^2}{\|u\|^2} \geq \frac{(\mathrm{Re}\langle Ku|u\rangle)^2}{\|u\|^2} = \frac{\left(\sum_k \|R_k u\|^2\right)^2}{4\,\|u\|^2},$$

which gives $\|R_k u\|^4 \leq 4\,\|u\|^2\,\|Ku\|^2$ and $D_{RN} \supset \mathrm{Dom}(K)$. Analogously, we get $D_{RN} \supset \mathrm{Dom}(K^*)$. Up to now we have proved conditions (i), (ii), (iv), (v), (vii).

To prove condition (iii) we have to show that the set D given in the Proposition is a core for K, K^*, a, $a^\dagger$, b, $b^\dagger$. By $\langle u|ae_{n,m}\rangle = \sqrt{n}\ \overline{u_{n-1,m}}$, we get

$$\{u \in \mathcal{H} :\ \exists \phi \in \mathcal{H} : \langle u|ae_{n,m}\rangle = \langle \phi|e_{n,m}\rangle\ \forall n, m\} = \mathrm{Dom}(a^\dagger).$$

Similar considerations hold for $a^\dagger$, b, $b^\dagger$ and D is a core for a, $a^\dagger$, b, $b^\dagger$. Analogously, by writing the expressions of $Ke_{n,m}$ e $K^*e_{n,m}$ we deduce that D is a core for K and K^*.

By (10) both K e K^* are *dissipative* [22, Definition 4.1 p. 13] and, by [22, Corollary 4.4 p. 15], they are generators of contraction semigroups in $\mathcal{H}$. This complete the proof of point (vi).

Finally, let us consider point (viii).

We have $\mathrm{Dom}\big(C^{1/2}\big) = \Big\{u \in \mathcal{H} : \sum_{n,m\geq 0} (n+2m)^4\, |u_{n,m}|^2 < +\infty\Big\}$, which is obviously contained in $\mathrm{Dom}(F) = \mathrm{Dom}(K)$ given in (45).

For any $\epsilon > 0$ take $D_\epsilon = D = \widetilde{D}$. Then, $C_\epsilon^{1/2} D_\epsilon \subset \widetilde{D}$ because C_ϵ is diagonal in the canonical basis. For large n and/or m, $C_\epsilon\, e_{n,m}$ goes as $1/\big(\epsilon(n+2m)^2\big)$ and each operator $F^*_{ij} C_\epsilon^{1/2}|_{D_\epsilon}$ is bounded (the worst case is for $F^*_{00} = K^*$). This is point (a).

By explicitly computing the left hand side of the inequality in point (b), we see that we have to prove the inequality

$$2\,\mathrm{Re} \sum_{i\geq 1} \left\langle u_i + \frac{1}{2}\, R_i u_0 \Big| [C_\epsilon, R_i] u_0 \right\rangle \leq \sum_{i\geq 0} \left(b_1 \langle u_i | C_\epsilon u_i\rangle + b_2\, \|u_i\|^2 \right).$$

The proof of this inequality is long and we give only a sketch.

Let us note that for any function of the operator N one has $af(N) = f(N+1)a$, $a^\dagger f(N+1) = f(N)a^\dagger$, $bf(N) = f(N+2)b$, $b^\dagger f(N+2) = f(N)b^\dagger$. We take $f(x) = \frac{x^4}{(1+\epsilon x^4)^2}$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, so that $f(N) = C_\epsilon$.

By these relations and standard estimates one can obtain

$$\mathrm{Re}\langle \alpha a u | [C_\epsilon, \alpha a] u\rangle \leq |\alpha|^2 \, \langle a^\dagger a u | \, |f(N-1) - f(N)| \, u\rangle .$$

By the relations above, expansion in the canonical basis and standard estimates we can prove also

$$2\,\mathrm{Re}\langle v | [C_\epsilon, \alpha a] u\rangle \leq |\alpha|^2 \, \langle a^\dagger a u | \, |f(N-1) - f(N)| \, u\rangle + \langle v | \, |f(N) - f(N+1)| \, v\rangle .$$

Analogous estimates can be obtained in the cases involving $a^\dagger$, b, $b^\dagger$. All together these results give

$$\begin{aligned} 2\,\mathrm{Re} \sum_{i\geq 1} \Big\langle u_i + \frac{1}{2}\, R_i u_0 \Big| [C_\epsilon, R_i] u_0 \Big\rangle &\leq \sum_{i=1,4,5} \big\langle u_i \big| \, |f(N+2) - f(N)| \, u_i \big\rangle \\ &+ 4\kappa \,(\overline{n}+1) \, \big\langle a^\dagger a u_0 \big| \, |f(N-1) - f(N)| \, u_0 \big\rangle + \big\langle a^\dagger a u_8 \big| \, |f(N-1) - f(N)| \, u_8 \big\rangle \\ &+ 4\kappa \overline{n} \Big\langle \big(a^\dagger a + 1\big) \, u_0 \Big| \, |f(N+1) - f(N)| \, u_0 \Big\rangle + \sum_{i=2,3,6} \big\langle u_i \big| \, |f(N+1) - f(N)| \, u_i \big\rangle \\ &+ 4\kappa_p \,(\overline{n}_p + 1) \, \big\langle b^\dagger b u_0 \big| \, |f(N-2) - f(N)| \, u_0 \big\rangle + \big\langle b^\dagger b u_7 \big| \, |f(N-2) - f(N)| \, u_7 \big\rangle \\ &+ 4\kappa_p \overline{n}_p \Big\langle \big(b^\dagger b + 1\big) \, u_0 \Big| \, |f(N+2) - f(N)| \, u_0 \Big\rangle . \end{aligned}$$

By using the specific form of f and $2(x-1) \geq x$ for $x \geq 2$ and $3(x-2) \geq x$ for $x \geq 3$, we get

$$\left| 1 - \frac{f(x+1)}{f(x)} \right| \leq \frac{15}{x}\,, \qquad \left| 1 - \frac{f(x+2)}{f(x)} \right| \leq \frac{80}{x}\,, \qquad \text{for } x \geq 1;$$

$$\left| \frac{f(x-1)}{f(x)} - 1 \right| \leq \frac{64}{x}\,, \quad \text{for } x \geq 2; \qquad \left| \frac{f(x-2)}{f(x)} - 1 \right| \leq \frac{648}{x}\,, \quad \text{for } x \geq 3.$$

These inequalities can slightly modified to include also the cases $x = 0, 1, 2$. Then, by further straightforward estimates, one gets

$$\begin{aligned} 2\,\mathrm{Re} \sum_{i\geq 1} \Big\langle u_i + \frac{1}{2}\, R_i u_0 \Big| [C_\epsilon, R_i] u_0 \Big\rangle &\leq 4\,(\kappa \overline{n} + 16 \kappa_p \overline{n}_p) \, \|u_0\|^2 + 324 \langle u_7 | C_\epsilon u_7 \rangle \\ &+ 64 \langle u_8 | C_\epsilon u_8 \rangle + \sum_{i=2,3,6} \Big(\|u_i\|^2 + 15 \langle u_i | C_\epsilon u_i \rangle \Big) + 16 \sum_{i=1,4,5} \Big(\|u_i\|^2 + 5 \langle u_i | C_\epsilon u_i \rangle \Big) \\ &+ 8\,(32\kappa(\overline{n}+1) + 162\kappa_p(\overline{n}_p + 1) + 15\kappa\overline{n} + 60\kappa_p\overline{n}_p) \, \langle u_0 | C_\epsilon u_0 \rangle . \end{aligned}$$

This ends the proof of the inequality. □

To use the number operator $N = a^\dagger a + 2 b^\dagger b$, which commutes with H_0, is suggested by [10, Chapter 3], where the conservativity property of the minimal quantum dynamical semigroup in a similar model is proved.

Proposition 14. *Hypothesis 2 holds for the model of this section.*

Proof. We prove the sufficient condition of Theorem 5 with $A = K^*$, $L_k = N_k^* = -R_k$. Let D be as in Theorem 13 and, on their maximal domains, let us introduce the operators

$$Q := wN + 2\kappa\overline{n} + 2\kappa_p\overline{n}_p\,, \qquad w := \max\left\{2\kappa\left(2\overline{n}+1\right),\ \kappa_p\left(2\overline{n}_p+1\right)\right\},$$

$$Z := 2\kappa\left(2\overline{n}+1\right)a^\dagger a + 2\kappa_p\left(2\overline{n}_p+1\right)b^\dagger b + 2\kappa\overline{n} + 2\kappa_p\overline{n}_p\,.$$

By defining also $v := \min\left\{2\kappa\left(2\overline{n}+1\right),\ \kappa_p\left(2\overline{n}_p+1\right)\right\}$, on D we have

$$0 \leq \frac{v}{w}\,Q + \frac{w-v}{w}\left(2\kappa\overline{n} + 2\kappa_p\overline{n}_p\right) \leq Z \leq Q.$$

In particular we get $\operatorname{Dom}(Q) = \operatorname{Dom}(Z) = \operatorname{Dom}(N) = \operatorname{Dom}(a^\dagger a) \cap \operatorname{Dom}(b^\dagger b)$, $D \subset \operatorname{Dom}(Q) \subset \operatorname{Dom}\left(Q^{1/2}\right) = \operatorname{Dom}\left(Z^{1/2}\right)$. The set D is a core for $Q^{1/2}$.

In the proof of Theorem 13 it is shown that $\operatorname{Dom}(K^*) \subset D_{RN}$. But one can check that $D_{RN} = \operatorname{Dom}\left(Q^{1/2}\right)$, so, we have $\operatorname{Dom}(K^*) \subset \operatorname{Dom}\left(Q^{1/2}\right) = \operatorname{Dom}\left(Z^{1/2}\right)$.

Finally, we get $\bigcap_{k\geq 1} R_k\big(\operatorname{Dom}(K^{*2})\big)$ by the fact that the R_ks are proportional to a, $a^\dagger$, b or $b^\dagger$ and that $\operatorname{Dom}(K^{*2}) \subset \operatorname{Dom}(a^\dagger a) \cap \operatorname{Dom}(b^\dagger)$, as one can check.

For $u \in D$ we get by direct computations

$$-2\operatorname{Re}\langle u|K^*u\rangle = \sum_{k\geq 1}\left\|R_k u\right\|^2 = \left\|Z^{1/2}u\right\|^2, \qquad \left\|Z^{1/2}u\right\| \leq \left\|Q^{1/2}u\right\|,$$

$$\left\|Q^{1/2}u\right\|^2 = w\left\|au\right\|^2 + 2w\left\|bu\right\|^2 + 2\left(\kappa\overline{n} + \kappa\overline{n}_p\right)\left\|u\right\|^2,$$

$$\begin{aligned} &2\operatorname{Re}\langle Q^{1/2}u|Q^{1/2}K^*u\rangle + \sum_{k\geq 1}\left\|Q^{1/2}R_k u\right\|^2 \\ &\qquad = 2\left(\kappa\overline{n} + \kappa\overline{n}_p\right)\left\|u\right\|^2 - 2w\kappa\left\|au\right\|^2 - 4w\kappa_p\left\|bu\right\|^2 \leq \left\|Q^{1/2}u\right\|^2. \end{aligned}$$

Then, these inequalities can be extended to the domains required in Theorem 5 and this ends the proof. $\square$

In order to describe the two photocounters and the homodyne detector we have to specialise the observables (22); what we need is to take $m = 3$ and [5]

$$X(\alpha,t) = \begin{cases} \Lambda_{\alpha\alpha}(t), & \alpha = 1,\,2, \\ \displaystyle\int_0^t \left(\mathrm{e}^{-\mathrm{i}(\theta_3-\omega_C t)}\,\mathrm{d}A_3(s) + \mathrm{e}^{\mathrm{i}(\theta_3-\omega_C t)}\,\mathrm{d}A_3^\dagger(s)\right), & \alpha = 3. \end{cases}$$

This means that the quantities in Hypothesis 3 are

$$c(t) = 0, \quad b(t) = 0, \quad B^1 = |z_1\rangle\langle z_1|, \quad B^2 = |z_2\rangle\langle z_2|, \quad B^3 = 0, \tag{47a}$$

$$h^1(t) = h^2(t) = 0, \quad h^3_i(t) = \delta_{i3}\, \mathrm{e}^{\mathrm{i}(\theta_3 - \omega_C t)}. \tag{47b}$$

This choice trivially satisfies Hypothesis 3 and the expressions of the quantities in Definition 6 become

$$\mathsf{S}\big(k(s)\big) = \mathbb{1} + \sum_{j=1}^{2} \Big(\mathrm{e}^{\mathrm{i}k_j(s)} - 1\Big)\, |z_j\rangle\langle z_j|, \qquad \mathsf{r}_t(k)(s) = 1_{(0,t)}(s)\, \mathrm{i}k_3(s) h^3(s)\,,$$

$$\big(\mathsf{S}_t(k)g\big)(s) = 1_{(0,t)}(s) \sum_{j=1}^{2} \Big(\mathrm{e}^{\mathrm{i}k_j(s)} - 1\Big)\, g_j(s)\, z_j + g(s), \qquad \mathsf{r}(k;s) = \mathrm{i}k_3(s) h^3(s).$$

Finally, in order to describe a coherent monochromatic laser pumping the b-mode as in the source term in the master equation (41), we have to take a coherent state of the field with f-function given by

$$f_i(t) = \delta_{i4}\, \frac{\mathrm{i}\lambda \mathrm{e}^{-2\mathrm{i}\omega_C t}}{\overline{\beta_2}}\, 1_{(0,T)}(t). \tag{48}$$

We are assuming $\beta_2 \neq 0$ and we understand that T is a large time (needed to have an L^2-function), but that $T \to +\infty$ in the reduced characteristic operator.

In conclusion the model just described is well defined, as it satisfies all the hypotheses introduced in this paper. Moreover, one can check that the associated formal master equation (Eq. (40) for $k = 0$) reduces to Eq. (41), as we wanted.

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